

EMBEDDINGS OF HOMOLOGY EQUIVALENT MANIFOLDS WITH BOUNDARY

D. GONÇALVES AND A. SKOPENKOV

ABSTRACT. We prove a theorem on equivariant maps implying the following two corollaries:

(1) Let N and M be compact orientable n -manifolds with boundaries such that $M \subset N$, the inclusion $M \rightarrow N$ induces an isomorphism in integral cohomology, both M and N have $(n-d-1)$ -dimensional spines and $m \geq \max\{n+3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\text{Emb}^m(N) \rightarrow \text{Emb}^m(M)$ is bijective. Here $\text{Emb}^m(X)$ is the set of embeddings $X \rightarrow \mathbb{R}^m$ up to isotopy (in the PL or smooth category).

(2) For a 3-manifold N with boundary whose integral homology groups are trivial and such that $N \not\cong D^3$ (or for its special 2-spine N) there exists an equivariant map $\tilde{N} \rightarrow S^2$, although N does not embed into \mathbb{R}^3 .

The second corollary completes the answer to the following question: for which pairs (m, n) for each n -polyhedron N the existence of an equivariant map $\tilde{N} \rightarrow S^{m-1}$ implies embeddability of N into \mathbb{R}^m ? An answer was known for each pair (m, n) except $(3, 3)$ and $(3, 2)$.

This note is on the classical problem of classification of embeddings into Euclidean spaces. For recent surveys see [Sk08, MA]; whenever possible we refer to these surveys not to original papers. As a main tool we use the Haefliger-Wu invariant defined below.

We begin with the formulation of our main homotopy result. Let $\tilde{N} = \{(x, y) \in N \times N \mid x \neq y\}$ be the *deleted product* of N . Let \mathbb{Z}_2 act on \tilde{N} and on S^{m-1} by exchanging factors and antipodes, respectively. Denote by $\pi_{eq}^{m-1}(\tilde{N})$ be the set of equivariant maps $\tilde{N} \rightarrow S^{m-1}$ up to equivariant homotopy. The set $\pi_{eq}^{m-1}(\tilde{N})$ can be effectively calculated [CF60, beginning of §2, Ad93, 7.1, Sk02, §6, Sk08, §5]. Note that $\pi_{eq}^{m-1}(\tilde{N}) = \emptyset$ for $m < n$ because $\tilde{N} \supset \tilde{D}^n \simeq_{eq} S^{n-1}$.

We omit \mathbb{Z} -coefficients from the notation.

Theorem. *Let N and M be compact orientable connected n -manifolds with non-empty boundaries such that $M \subset N$ and the inclusion $M \rightarrow N$ induces an isomorphism in cohomology. Then the restriction-induced map $\pi_{eq}^{m-1}(\tilde{N}) \rightarrow \pi_{eq}^{m-1}(\tilde{M})$ is bijective.*

This homotopy result is interesting because of the following topological corollaries. Denote $\text{CAT} = \text{DIFF}$ or PL . For a CAT manifold N let $\text{Emb}_{CAT}^m(N)$ be the set of CAT embeddings $N \rightarrow \mathbb{R}^m$ up to CAT isotopy. A folklore general conjecture, supported by some known results (for a survey see e.g. [RS99]) is that $\text{Emb}_{CAT}^m(N)$ is not changed under homology equivalence of N (i.e. under a map $f : M \rightarrow N$ between manifolds inducing an isomorphism in (co)homology), in the PL case for $m \geq n+3$ and in the DIFF case for $m \geq \frac{3n}{2} + 2$.

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Corollary. *Let N and M be compact orientable n -manifolds with non-empty boundaries such that $M \subset N$, the inclusion $M \rightarrow N$ induces an isomorphism in cohomology, both M and N have $(n - d - 1)$ -dimensional spines and $m \geq \max\{n + 3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\text{Emb}_{CAT}^m(N) \rightarrow \text{Emb}_{CAT}^m(M)$ is bijective.*

Recall that

- a subpolyhedron K of a manifold N is called a *spine* of N if N is a regular neighborhood of K in N (or, equivalently, if N collapses to K) [RS72].¹

- a closed manifold N (or a pair $(N, \partial N)$) is called *homologically d -connected*, if N is connected and $H_i(N) = 0$ for each $i = 1, \dots, d$ (or $H_i(N, \partial N) = 0$ for each $i = 0, \dots, d$).

In the DIFF category the restriction $m \geq \frac{3n+2-d}{2}$ can be relaxed to $m \geq \frac{3n+1-d}{2}$.

By the Corollary, any homology ball unknots in codimension at least 3, cf. [Sc77].

The Corollary follows from the Theorem and the bijectivity of α -invariant [RS99, §4, Sk02, Theorems 1.1 $\alpha\partial$ and 1.3 $\alpha\partial$], which is defined as follows. For an embedding $f : N \rightarrow \mathbb{R}^m$ define a map

$$\tilde{f} : \tilde{N} \rightarrow S^{m-1} \quad \text{by} \quad \tilde{f}(x, y) = \frac{fx - fy}{|fx - fy|}.$$

The equivariant homotopy class $\alpha(f)$ of the above-defined \tilde{f} in $\pi_{eq}^{m-1}(\tilde{N})$ is clearly an isotopy invariant. Thus is defined the *Haefliger-Wu (deleted product) invariant*

$$\alpha = \alpha_{CAT}^m(N) : \text{Emb}_{CAT}^m(N) \rightarrow \pi_{eq}^{m-1}(\tilde{N}).$$

Remarks. (a) If in Theorem and in Corollary the inclusion-induced homomorphism $H^i(N) \rightarrow H^i(M)$ is an isomorphism only for $i \geq l > 0$, then the corresponding restriction-induced maps are bijective for $m \geq n + l$ and surjective for $m = n + l - 1$.

(b) The assumption that (N, M) is a codimension 0 pair is essential in the Theorem and the Corollary. Indeed, take $N = D^p \times S^q$ and $M = S^q$. For $m \geq 3q/2 + 2$ we have $\#\text{Emb}^m(S^q) = 1$ while $\text{Emb}^m(D^p \times S^q) = \pi_q(V_{m-q,p})$ can contain more than one element (specific examples are particularly easy to find for $p = 1$, when $V_{m-q,p} \simeq S^{m-q-1}$).

(c) The assumption that N has boundary is not essential in the Theorem and the Corollary. But these results are trivial for closed N : if N is closed and $M \neq N$, then the assumptions are never fulfilled because $H^n(N) \not\cong H^n(M)$.

(d) The conclusion of the Theorem for closed manifolds is not always fulfilled, because there are closed manifolds non-embeddable in the same dimension as the corresponding punctured manifolds.

(e) The Theorem is clearly true for $m < n$ because both sets are empty. We conjecture that the Theorem holds for $m = n$ and for $m = n + 1$.

Now let us present motivation for the second corollary of the Theorem. From the construction of the map \tilde{f} above it follows that

(*) if N embeds into \mathbb{R}^m , then there exists an equivariant map $\tilde{N} \rightarrow S^{m-1}$.

The existence of an equivariant map $\tilde{N} \rightarrow S^{m-1}$ can be checked for many cases [CF60, beginning of §2, Ad93, 7.1, Sk08, §5]. Thus if a converse to (*) is true, the embedding problem is reduced to a manageable (although not trivial) algebraic problem. So in 1960s

¹We remark that for a compact connected n -manifold N with boundary, the property of having an $(n - d - 1)$ -dimensional spine is close to d -connectedness of $(N, \partial N)$. Indeed, for a compact connected n -manifold N with boundary and an $(n - d - 1)$ -dimensional spine, the pair $(N, \partial N)$ is homologically d -connected. On the other hand, every compact connected n -manifold N with boundary for which $(N, \partial N)$ is d -connected, $\pi_1(\partial N) = 0$, $d + 3 \leq n$ and $(n, d) \notin \{(5, 2), (4, 1)\}$, has an $(n - d - 1)$ -dimensional spine [Wa64, Theorem 5.5, Ho69, Lemma 5.1 and Remark 5.2].

there appeared a problem to find conditions under which the converse to (*) is true. The converse for (*) was known to be

true for an n -polyhedron N and $2m \geq 3n + 3$ or $m = 2n = 2$ [RS99, §4, Sk08, §5], cf. [Sk98, Theorem 1.3];

false for each pair (m, n) such that $\max\{4, n\} \leq m \leq \frac{3n}{2} + 1$ and some n -polyhedron N [RS99, §4, Sk08, §5], cf. [Sk98, Example 1.4].

In the only remaining cases $m = 3$ and $n \in \{2, 3\}$ it was unknown if the converse to (*) is true. The counterexamples to the converse of (*) for $m = n \geq 4$ and $m = n + 1 \geq 4$ [MS67, Hu88] cannot be directly extended to $m = 3$ because they used m -dimensional contractible manifold distinct from the m -ball, which apparently does not exist for $m = 3$.

Recall that a *homology n -ball* is an n -manifold with boundary whose homology groups are the same as those of the n -ball. A *special spine* is defined e.g. in [Ca65].

Proposition. *The converse to (*) is false in the cases $m = 3$ and $n \in \{2, 3\}$: if N is either a non-trivial homology ball or a special spine of a non-trivial homology ball, then N does not embed into \mathbb{R}^3 but there exists an equivariant map $\tilde{N} \rightarrow S^2$.*

Proof. The non-embeddability follows because if a special spine of a homology ball N embeds into \mathbb{R}^3 , then the regular neighborhood in \mathbb{R}^3 of this spine is homeomorphic to N [Ca65], which contradicts to the non-triviality of N .

It suffices to prove the existence of an equivariant map $\tilde{N} \rightarrow S^2$ for a homology 3-ball N .² Analogously to [Ad93, end of §7.1] (or by Lemma 2 below) it suffices to prove that $H^i(\tilde{N}) = 0$ for each $i \geq 3$. We prove this for $i = 3$; the proof for each $i \geq 4$ is analogous. Let Δ be the interior of a closed regular neighborhood in $N \times N$ of the diagonal. Then

$$\begin{aligned} H^3(\tilde{N}) &\cong H^3(N \times N - \Delta) \cong H_3(N \times N - \Delta, \partial(N \times N - \Delta)) \cong \\ &\cong H_3(N \times N, \text{Cl } \Delta \cup \partial(N \times N)) \cong H_2(\text{Cl } \Delta \cup \partial(N \times N)) = 0, \quad \text{where} \end{aligned}$$

- the first isomorphism follows because $N \times N - \Delta$ is a deformation retract of \tilde{N} ,
- the second one by Lefschetz duality (recall that \tilde{N} is orientable if N is a homology ball),
- the third one by excision,
- and the fourth one by exact sequence of pair.

Using the Mayer-Vietoris sequence for

$$\partial(N \times N) = N \times \partial N \bigcup_{\partial N \times \partial N} \partial N \times N$$

and noting that $\partial N \cong S^2$, we prove that $H_2(\partial(N \times N)) = 0$. Using the Mayer-Vietoris sequence for $\text{Cl } \Delta \cup \partial(N \times N)$ and noting that $\Delta \simeq N$ and $\text{Cl } \Delta \cap \partial(N \times N)$ is a regular neighborhood in $N \times N$ of the diagonal of ∂N , i.e. is homotopy equivalent to $\partial N \cong S^2$, we prove the last isomorphism. \square^3

Manifolds \tilde{N} and \tilde{M} are homotopy equivalent to $(2n - 1)$ -dimensional CW complexes. Hence the Theorem follows by the cases $l = 0$ of Lemmas 1 and 2, see below. Remark (a) follows by (the general cases of) Lemma 1 and 2.

²This existence follows from the (unproved) case $m = n = 3$ of the Theorem because an inclusion of the standard ball into N induces isomorphisms in cohomology.

³Another proof of the Proposition could possibly be obtained by using the fact that for the homology 3-ball N , which is a punctured boundary of the Mazur 4-manifold, there exists an equivariant map $\Sigma \tilde{N} \rightarrow S^3$ [MRS03]. The obstruction to equivariant desuspension of this map on \tilde{P} (where P is the special spine of N) apparently lies in $H^4(\tilde{P})$, which group is trivial because P is acyclic [We68].

Lemma 1. *Let N and M be compact orientable connected n -manifolds with non-empty boundaries such that $M \subset N$ and the inclusion induces an isomorphism $H^i(N) \rightarrow H^i(M)$ for $i \geq l$. Then $H^i(\tilde{N}, \tilde{M}) = 0$ for each $i \geq n + l$.*

Lemma 2. [BG71, 3.2] *Suppose X, Y are finite connected CW-complexes with free involutions, $f : X \rightarrow Y$ is an equivariant map and l is a non-negative integer. If $f^* : H^i(Y) \rightarrow H^i(X)$ is an isomorphism for each $i > l$ and is onto for $i = l$, then*

(d_l) $f^\# : \pi_{eq}^i(Y) \rightarrow \pi_{eq}^i(X)$ is a 1-1 correspondence for $i > l$ and is onto for $i = l$.

We give a proof of Lemma 2 (which was not presented in [BG71]) using standard argument and following [HH62, pp. 236-237], cf. [Me09, Proof of Lemma 8.1]. Lemma 2 was used in the previous version [GS06] of this paper; the proof was essentially presented there but contains mistakes which are corrected here.

Proof of Lemma 1 for $l = 0$. Let N_0 and M_0 be the interiors of N and M , respectively. It suffices to prove Lemma 1 for N and M replaced by N_0 and M_0 .

(Indeed, the collaring theorem for the boundary of a manifold states that *there is a neighborhood of ∂M in M which is homeomorphic to the product $\partial M \times [0, 1)$ so that $\partial M \times \{0\}$ is mapped homeomorphically to the boundary*. Therefore there is an embedding $\phi : N \rightarrow N_0$ which is a homotopy inverse of the inclusion $N_0 \rightarrow N$. Analogously $\phi \times \phi : \tilde{N} \rightarrow \tilde{N}_0$ is a homotopy inverse of the inclusion $\tilde{N}_0 \rightarrow \tilde{N}$. Same observations hold for N replaced by M . So it suffices to prove Lemma 1 for N and M replaced by N_0 and M_0 .)

Let $x_0 \in M_0 \subset N_0$ be a base point for M_0 and N_0 . Consider the following mapping of bundles (which are given by projections onto the first factor):

$$\begin{array}{ccccc} M_0 - x_0 & \rightarrow & \tilde{M}_0 & \rightarrow & M_0 \\ \downarrow \subset & & \downarrow \subset & & \downarrow \subset \\ N_0 - x_0 & \rightarrow & \tilde{N}_0 & \rightarrow & N_0 \end{array}$$

The action of $\pi_1(M_0)$ in the cohomology $H^i(M_0 - x_0)$ of the fiber is trivial for each i .

(Indeed, this follows for $i = n$ because $H^n(M_0 - x_0) = 0$ and for $i < n - 1$ because $H^i(M_0 - x_0) \cong H^i(M_0)$ and the bundle is the restriction of the trivial bundle $M_0 \times M_0 \rightarrow M_0$. For $i = n - 1$ we have $M_0 - x_0 \simeq M_0 \vee S^{n-1}$, so $H^{n-1}(M_0 - x_0) \cong H^{n-1}(M_0) \oplus \mathbb{Z}$. The action of an element $\alpha \in \pi_1(M_0)$ is given by the identity on the first summand and multiplication by the sign of the loop on \mathbb{Z} . Since M_0 is orientable, the action is identical.)

The same holds for the second bundle, where M is replaced by N .

By excision the inclusion of the pairs $(M_0, M_0 - x_0) \rightarrow (N_0, N_0 - x_0)$ induces an isomorphism in cohomology.

Proof of Lemma 1: completion for $l = 0$. Applying 5-lemma for the inclusion-induced mapping of exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \rightarrow N_0 - x_0$ induces an isomorphism in cohomology. Hence using the triviality of the action and the Universal Coefficients Theorem we obtain that the restriction induces an isomorphism

$$r : H^p(N_0; H^q(N_0 - x_0)) \rightarrow H^p(M_0; H^q(M_0 - x_0)) \quad \text{for each } p, q.$$

This r is a homomorphism of the E_2 -terms of the Leray-Serre cohomology spectral sequences of the above bundles. By the Zeeman Comparison Theorem of spectral sequences [Ze57], the restriction $H^i(\tilde{N}_0) \rightarrow H^i(\tilde{M}_0)$ is an isomorphism for each i . This implies Lemma 1. \square^4

⁴A statement on cohomology of compact manifolds should have a proof involving only cohomology of compact manifolds (recall that we may assume that $\tilde{N} = \tilde{N}_\varepsilon$ is compact). The above proof has such an interpretation in terms of only compact spaces. Lemma 1 can also be proved analogously to proof of the Proposition above.

Proof of Lemma 1: completion for the general case. Applying the 5-lemma for the inclusion-induced mapping of exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \rightarrow N_0 - x_0$ induces an isomorphism in H^i for $i \geq l$. Hence using the triviality of the action and the Universal Coefficients Theorem we obtain that the restriction induces an isomorphism

$$r : H^p(N_0; H^q(N_0 - x_0)) \rightarrow H^p(M_0; H^q(M_0 - x_0)) \quad \text{for } p + q \geq n + l - 1.$$

Hence r is an isomorphism of for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. This r is a homomorphism of the E_2 -terms of the Leray-Serre cohomology spectral sequences of the above bundles. Now using standard argument of homological algebra as in the Zeeman Comparison Theorem of spectral sequences [Ze57] we obtain that the restriction-induced homomorphism between $E_r^{p,q}$ terms is an isomorphism for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. Since $E_{n-l} = E_{n-l+1} = \dots = E_\infty$, the restriction induces on E_∞ terms an isomorphism for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. Hence the restriction $H^i(\tilde{N}_0) \rightarrow H^i(\tilde{M}_0)$ is an isomorphism for each $i \geq n + l$ and an epimorphism for $i = n + l - 1$. Therefore by the exact sequence of pair $H^i(\tilde{N}_0, \tilde{M}_0) = 0$ for each $i \geq n + l$. Hence $H^i(\tilde{N}, \tilde{M}) = 0$ for each $i \geq n + l$. \square

Proof of Lemma 2. We may assume that $f : X \rightarrow Y$ is an inclusion. Consider the following assertion:

(c_l) $H^i(Y', X'; G_\varphi) = 0$ for each $i > l$, finitely-generated abelian group G , involution $\varphi : G \rightarrow G$ and local coefficient system G_φ associated to φ and double cover $(Y, X) \rightarrow (Y', X')$.

(Local coefficient system G_φ is defined by the following action of $\pi_1(Y')$ on G . Take a representative $\alpha' : [0, 1] \rightarrow Y'$, $\alpha'(0) = \alpha'(1)$, of $[\alpha'] \in \pi_1(Y')$. Take a lift $\alpha : [0, 1] \rightarrow Y$ of α' . If $\alpha(0) = \alpha(1)$, then $[\alpha']$ acts identically on G . If $\alpha(0) \neq \alpha(1)$, then $[\alpha']$ acts as φ . Clearly, this action is well-defined.)

Since Y is finite-dimensional,⁵ (c_l) holds for large enough l . Consider the following part of the Smith-Richardson-Thom-Gysin sequences associated to the double cover $(Y, X) \rightarrow (Y', X')$ (see the Smith-Richardson-Thom-Gysin Sequence Theorem below):

$$0 = H^i(Y, X; G) \rightarrow H^i(Y', X'; G_\varphi) \rightarrow H^{i+1}(Y', X'; G_{-\varphi}).$$

By the hypothesis of Lemma 2 $H^i(Y, X) = 0$ for each $i > l$. So by the Universal Coefficients Formula $H^i(Y, X; G) = 0$ for each $i > l$. Then by downward induction on l we obtain (c_l).

Denote by a the involution on $\pi_k(S^i)$ induced by the antipodal involution on S^i .⁶ The obstructions to extension to Y of an equivariant map $X \rightarrow S^i$, and to homotopy uniqueness of such an extension, assume values in $H^{k+1}(Y', X'; \pi_k(S^i)_a)$ and $H^k(Y', X'; \pi_k(S^i)_a)$.⁷ These groups are trivial for $k < i$ because $\pi_k(S^i) = 0$, and for $k \geq i > l$ by (c_l). So (d_l) holds. \square

For a reader's convenience we present the following slight and possibly known extension of the Smith-Richardson-Thom-Gysin sequence. Cf. [Me09, arxiv v4, Remark 2.3 and p.9, lines 14-25].

⁵It would be interesting to know if Lemma 2 holds for infinite-dimensional complexes. Note that it does hold for infinite-dimensional complexes $S^{l-1} \rightarrow S^\infty$.

⁶Note that $a = \text{id}$ for i odd and $a = -\text{id}$ for i even and $k \leq 2i - 2$.

⁷This can be deduced either from obstruction theory for extension of maps with non-simply-connected range $\mathbb{R}P^\infty$ [HW60] or analogously to [CF60, beginning of §2, Ad93, 7.1] as follows. Denote by t the involution

on Y and its restriction to X . Define a bundle $g : \frac{Y \times S^i}{(x, s) \sim (tx, -s)} \xrightarrow{S^i} Y'$ by $g[(x, s)] = [x]$. Equivariant maps $Y \rightarrow S^i$ up to equivariant homotopy are in 1-1 correspondence with cross-sections of g up to equivalence. So the required obstructions are obstructions to

(*) extendability of a section on X' to a section on Y' for each $i \geq l$, and to

(**) uniqueness of such an extension (up to equivalence) for $i > l$.

The action of $\pi_1(Y')$ on homotopy groups of the fiber S^i gives rise to local coefficient system $\pi_k(S^i)_a$.

Smith-Richardson-Thom-Gysin Sequence Theorem. *Let X' be a connected space, $X \rightarrow X'$ a double covering and G a module with an involution φ . Consider the local coefficient system G_φ on X' associated to the double covering and φ . Then there is a long exact sequence*

$$\cdots \rightarrow H^{p-1}(X'; G_\varphi) \rightarrow H^p(X'; G_{-\varphi}) \rightarrow H^p(X; G) \rightarrow H^p(X'; G_\varphi) \rightarrow H^{p+1}(X'; G_{-\varphi}) \rightarrow \cdots$$

If 2 is invertible in G (in particular, if either $G = \mathbb{Q}$ or $G = \mathbb{Z}_p$ for p an odd prime), then we have splittable short exact sequence

$$0 \rightarrow H^p(X'; G_{-\varphi}) \rightarrow H^p(X; G) \rightarrow H^p(X'; G_\varphi) \rightarrow 0 \quad \text{so that}$$

$$H^p(X; G) \cong H^p(X'; G_{-\varphi}) \oplus H^p(X'; G_\varphi).$$

If $G = \mathbb{Z}$ and $\varphi = \text{id}$, then we get long exact sequence

$$\cdots \rightarrow H^{p-1}(X') \rightarrow H^p(X'; \mathbb{Z}_{-\text{id}}) \rightarrow H^p(X) \rightarrow H^p(X') \rightarrow H^{p+1}(X'; \mathbb{Z}_{-\text{id}}) \rightarrow \cdots$$

If $G = \mathbb{Z}$ and $\varphi = -\text{id}$, then we get long exact sequence

$$\cdots \rightarrow H^{p-1}(X'; \mathbb{Z}_{-\text{id}}) \rightarrow H^p(X') \rightarrow H^p(X) \rightarrow H^p(X'; \mathbb{Z}_{-\text{id}}) \rightarrow H^{p+1}(X') \rightarrow \cdots$$

Proof. Consider the fibration $F \rightarrow X \rightarrow X'$ which is the double covering, where F is a two-points set. For the spectral sequence with local coefficients [Si97, Theorem 2.9] we have $E_2^{p,q} = H^p(X', H^q(F; G)_\tau)$, where the coefficients are twisted according to double cover $X \rightarrow X'$ and the following involution τ of $H^q(F; G)$:

- $H^q(F; G) = 0$ and τ is trivial for $q > 0$, and
- $H^0(F; G) \cong G \oplus G$ and $\tau(a, b) := (\varphi(b), \varphi(a))$.

Then the spectral sequence contains at most one non-vanishing line. Hence⁸

$$H^p(X; G) \cong E_\infty^{p,0} \cong E_2^{p,0} \cong H^p(X', (G \oplus G)_\tau).$$

Let $H = \{(m, -m) \in G \oplus G \mid m \in G\}$. We have $\tau(m, -m) = (\varphi(-m), \varphi(m)) = (-\varphi(m), \varphi(m))$. Hence $\tau(H) = H$ and $(H, \tau|_H) \cong (G, -\varphi)$. Then $(G \oplus G)/H$ has ‘the quotient’ involution τ/H . Clearly, $((G \oplus G)/H, \tau/H) \cong (G, \varphi)$. Now the first part of the theorem follows from the cohomological long exact sequence associated with the short exact sequence of twisted coefficients $(H, \tau|_H) \rightarrow (G \oplus G, \tau) \rightarrow ((G \oplus G)/H, \tau/H)$.

The ‘further’ part where 2 is invertible follows from the fact that the above short exact sequence splits: the homomorphism $s : (G \oplus G)/H \cong G \rightarrow G \oplus G$ defined by $s(m) = (m/2, m/2)$ respects involutions and is a splitting. The ‘further’ part where $G = \mathbb{Z}$ is clear. \square

⁸The twisting of $H^q(F; G)$ is as required by [Si97, 2.7]. Note that [Si97, 2.8] is not required for the statement of [Si97, Theorem 2.9] (but is required for the proof). Note that the purpose of [Si97, Theorem 2.9] was to calculate cohomology of the total space of a fibration not with any non-twisted coefficient system but with the twisted coefficient system coming from a twisted coefficient system in the base.

The isomorphism $H^p(X; G) \cong H^p(X', (G \oplus G)_\tau)$ has two simpler proofs not involving spectral sequences. According to S. Melikhov, it follows easily from definitions, as explained in [Ha, Example 3.H.2] (the case of arbitrary φ follows from the case $\varphi = \text{id}$ because the involution $(a, b) \mapsto (\varphi(b), \varphi(a))$ is obtained from the involution $(a, b) \mapsto (b, a)$ by an automorphism of $G \oplus G$ [Br82, Corollary III.5.7]), or, alternatively, is a special case of the Vietoris Mapping Theorem, [Br97, Theorem 11.1].

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DEPARTAMENTO DE MATEMÁTICA, IME, UNIVERSITY OF SÃO PAULO, CAIXA POSTAL 66281, AGÊNCIA CIDADE DE SÃO PAULO 05311-970, SÃO PAULO, SP, BRASIL. E-MAIL: DLGONCAL@IME.USP.BR

INDEPENDENT UNIVERSITY OF MOSCOW, B. VLASYEVSKIY, 11, 119002, MOSCOW, RUSSIA. E-MAIL: SKOPENKO@MCCME.RU